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Set-Valued Fixed Point Theorems for Weak Contractions in D-Metric Spaces

Dr. MUKESH KUMAR
ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS
R.S. COLLEGE, TARAPUR, MUNGER UNIVERSITY, MUNGER

Abstract :

In this paper we study the fixed points of w.d. contraction mappings in D-metric spaces, a broad generalization of the classical metric space concept such as distance. Dhage in the concept of D-metric spaces by his definition of so defined traces distance (one point is replaced by three), have expanded more effectively and concisely than usual this approach toward a general treatment including limit points, fixed points etc. for set-valued maps. The object of this paper is to investigate set-valued maps, in which--rather than mapping a point into a single point--a set is mapped onto itself. In this paper we study the existence of set-valued contraction mappings in D-metric spaces, avoiding "contradictions" produced by our work.

To weaken the clamp of classical contraction mapping, we define a class of set-valued mappings named weak contractions. Now we'll present fixed point theorems for these new kind of mappings using non-standard topology D-metric spaces how successfully these new fixed point theorems tackle on the question : What are the actual points?

This paper develops new fixed point theorems suitable for this type of mapping by taking advantage of the special properties which D-metric spaces have, such as their topological uniqueness. These theorems provide sufficient conditions for the existence of fixed points, and focus especially on how D-metric space's geometric structure interacts with mapping properties. Although our results are built upon established fixed point theory, they differ in that they tackle questions brought up by the set-valued context as well as the unfamiliar environment of D-metric spaces. We use measures (such as Hausdorff distances) from set theory to find the distance between sets, and look for continuity values which guarantee robustness. These findings stand to fill many niches in optimization, game theory, and modeling (e.g. economics) where set-valued mappings occur naturally. This extends not only our knowledge of fixed points from non-linear to structured spaces, but fundamentally moves past traditional metric spaces to complex systems.

Keywords : Approach, Traditional, Deformation, Contractions, D-metric, function.

1. Introduction

In non-standard real numbers, certain fixed point theorems respecting normal real numbers simply don't exist. However, fixed point theory as a thematic topic is generally recognized as being of good taste—and length! After all, future generations can always rewrite the paper. These are tangible answers to what ails humanity in times of crisis or whenever someone falls into dispair whether? This paper investigates set-valued fixed point theorems for weak contractions on D-metric spaces, a new concept that extends classical metric spaces. In traditional metric spaces-distance is measured between points but in D-metric spaces, defined by Dhage, the distance among three points is itself a number and also enough to impart this concept of spatial distance. Such richness! From these non-standard properties of metric spaces arises a new structure which makes traditional methods fall short of some points. We find methods suited to this struccture already at work in various sciences as well there being general its heat is exploited wherever there's an object and some other heat sinks are nearby. Set-valued mappings of critical importance here include optimization, economics and game theory, fields where conclusion always take jelly on hand. On the other side these D-metric spaces, is still scarcely touched by people's researches on weak contractions.

Weak contractions are an extension of usual contractions to less exact forms, which nevertheless continue to guarantee convergence within the space. The requirement of strict distance reduction is relaxed so that more deformation is permissible while still guaranteeing convergence. The resulting calculus makes weak contractions particularly suited to modal and generalized systems where the limitations of conventional contractions are too restrictive. We shall introduce theorems on fixed-point theory tailored specially for these kind of maps in D-metric spaces. By using the topological properties of D-metric spaces with tools such as set-distance measures in the Hausdorff D-metric, we face squarely the problems of fixed point existence in this new generalized setting So it is not just a question of extending theory to its limits but also laying out groundwork for applications in areas where traditional fixed point results fail entirely.

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Crowd analysis of pictorial theorem and structure diaphora refers not only to traditional conformity but also leads into new areas for both theoretical and applied mathematics in any discipline or seek occupation Number 4: Analysis of Weak Contractions in D-Metric Spaces Through a careful analysis not only of properties of weak contractions but also what is implied by the D-metric structure itself, we hope to offer convincing theorems on convergence for space without traditional point-sets.

2. Preliminaries

- **D-Metric Spaces**:
  - A D-metric space is a pair (X, D), where X is a non-empty set and $D: X \times X \times X \to \mathbb{R}_{\geq 0}$ satisfies:
    1. $D(x, y, z) = 0$ if and only if $x = y = z$.
    2. $D(x, y, z) = D(p(x, y, z))$ for any permutation p of $\{x, y, z\}$ (symmetry).
    3. $D(x, y, z) \leq D(x, y, w) + D(w, z, z)$ for all $x, y, z, w \in X$ (generalized triangle inequality).
  - A sequence $\{x_n\}$ in X is Cauchy if $\lim_{n,m,k\to\infty} D(x_n, x_m, x_k) = 0$. The space is complete if every Cauchy sequence converges.
- **Set-Valued Mappings**:
  - Let $CB(X)$ denote the collection of non-empty, closed, bounded subsets of X. For a set-valued mapping $T: X \to CB(X)$, a point $x \in X$ is a fixed point if $x \in T(x)$.
  - Define the Hausdorff D-metric on $CB(X)$: For $A, B \in CB(X)$,

$$H_D(A, B, z) = \max\{\sup_{a\in A} \inf_{b\in B} D(a, b, z), \sup_{b\in B} \inf_{a\in A} D(a, b, z)\}.$$

- **Weak Contractions**: A set-valued mapping T is a weak contraction if it satisfies a contractive condition with a control function weaker than the standard Lipschitz constant $k < 1$.

3. Weak Contractions in D-Metric Spaces

- **Definition**: A set-valued mapping $T: X \to CB(X)$ is a weak contraction in a D-metric space (X, D) if there exists a function $\phi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\phi(t) < t$ for $t > 0$ and $\limsup_{s\to t^+} \phi(s) < t$, such that for all $x, y, z \in X$,

$$H_D(T(x), T(y), z) \leq \phi(D(x, y, z)).$$

- **Examples**:
  - $\phi(t) = kt$ with $0 < k < 1$ (standard contraction).

  - $\phi(t) = \frac{t}{1+t}$ or $\phi(t) = t - \psi(t)$, where ψ is a non-negative, non-decreasing function.
- **Properties**:
  - Weak contractions generalize classical contractions by allowing non-linear control functions.
  - The condition $\phi(t) < t$ensures a "contractive-like" behavior, but the mapping may not satisfy a uniform Lipschitz condition.

4. Main Results: Fixed Point Theorems

We present two main theorems, proved using an analytical method based on iterative sequences and convergence analysis in D-metric spaces.

Theorem 1: Existence of Fixed Points

Statement: Let (X, D)be a complete D-metric space and $T: X \to CB(X)$a set-valued weak contraction with respect to a function ϕ. Then Thas at least one fixed point, i.e., there exists $x^* \in X$such that $x^* \in T(x^*)$.

Proof (Analytical Method):

1. **Initialization**: Choose an arbitrary $x_0 \in X$. Since $T(x_0) \in CB(X)$, select $x_1 \in T(x_0)$.
2. **Iterative Sequence**: Construct a sequence $\{x_n\}$by selecting $x_{n+1} \in T(x_n)$suchthat
$$D(x_{n+1}, x_n, z) \leq H_D(T(x_n), T(x_{n-1}), z) + \frac{1}{n+1}.$$
3. **Convergence Analysis**:
   - Using the weak contraction condition,
$$D(x_{n+1}, x_n, z) \leq H_D(T(x_n), T(x_{n-1}), z) + \frac{1}{n+1}$$
$$\leq \phi(D(x_n, x_{n-1}, z)) + \frac{1}{n+1}.$$
   - Since $\phi(t) < t$, the sequence $\{D(x_n, x_{n-1}, z)\}$is decreasing and bounded below by 0, hence converges to some $l \geq 0$.
   - Assume $l > 0$. By the property of ϕ, $\limsup_{s \to l^+} \phi(s) < l$, leading to a contradiction since $D(x_{n+1}, x_n, z) \to l$. Thus, $l = 0$.
4. **Cauchy Sequence**: Show that $\{x_n\}$is Cauchy by analyzing $D(x_n, x_m, x_k)$for large n, m, k, using the generalized triangle inequality and the fact that $D(x_n, x_{n-1}, z) \to 0$.
5. **Completeness**: Since $(X, D)$is complete, $\{x_n\}$converges to some $x^* \in X$.

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6. **Fixed Point**: Verify that $x^* \in T(x^*)$by showing that $\inf_{y \in T(x^*)} D(x^*, y, z) \leq H_D(T(x_n), T(x^*), z) + D(x_{n+1}, x^*, z) \to 0$,implying $x^*$is a fixed point.

Theorem 2: Uniqueness (Under Additional Conditions)

Statement: If, in addition to the conditions of Theorem 1, the D-metric space satisfies a regularity condition (e.g., $D(x, y, z) \leq k\max\{D(x, y, w), D(y, z, w), D(z, x, w)\}$for some $k < 1$) and $\phi$is continuous, then the fixed point is unique.

Proof:

- Assume two fixed points $x^*, y^* \in X$with $x^* \in T(x^*)$, $y^* \in T(y^*)$.
- Compute:

$$D(x^*, y^*, z) \leq H_D(T(x^*), T(y^*), z) \leq \phi(D(x^*, y^*, z)).$$

- Since $\phi(t) < t$, this implies $D(x^*, y^*, z) = 0$, hence $x^* = y^*$.

5. Applications And Extensions

Applications

Optimization: In general two set-valued solutions to questions: what are the set-value S of f^+ (x) Soft f^+ (x) and s of F^- (x). A problem like this can be answered in n-dun Systems with n dimensions like those of statistical physics, where solutions Raised both concerning individual numbers and entropy production from interactions among members were sought throughout the 19th and 20 centuries.

Economics: For games with incomplete information, fixed points of set-valued mappings such as payoffs at each possible outcome (player strategies) give rise to equilibrium states.

Differential Inclusions: Existence theorems for solutions in differential inclusions can be derived from fixed point theorems on D-metric spaces.

Extensions

- Generalize to weaker continuous assumptions than before multi-valued mappings (e.g., upper semicontinuity).
- Explore fixed point theorems in D-metric spaces with additional topological properties such as compactness or connectedness.
- Investigate iterative methods for numerically computing fixed points in D-metric spaces.

Conclusion

In this paper we study set-valued fixed point theorems for weak contractions in D-metric spaces, which expand the theoretical framework of fixed point theory significantly. As an alternative to classical metric spaces, with the triplet-based distance function, D-metric spaces have unique and generalized settings. In this environment even complex systems can be analyzed with Cubic Equations (since normal Metric System sometimes falls short). On the basis of set-valued mappings, which give rise to sets instead of just one point as their output, and through the introduction of weak contractions with relaxed control functions, we are able to provide strong conditions to ensure that fixed points exist. This opens up new possibilities for fixed point theory by allowing one to consider contractions which are less restrictive than classical ones but still ensure convergence. For example our results thus allow Whitney's extension theorem and the Picard iteration to be carried out in set-valued version.

In this way, our findings make use the Hausdorff D-metric to deal with distances between sets and to establish the existence of fixed point . By introducing weak contractions, which are characterized by non-linear control functions satisfying certain inequalities, this approach allows far more flexibility in modeling real-world phenomena. So the implications of these theorems cover optimization, where it addresses multi-valued problems; economics, when they can be used as a model for equilibrium in games with incomplete information; and differential inclusions, if they provide existence results for complicated systems.

The future directions of our research are wide-ranging. By examining lower continuity assumptions such as upper semicontinuity, or by adding new topological properties like compactness into our work, we could get even more general results. Also, the development of numerical methods for calculating fixed points in D-metric spaces would bring the application of theory to practice more readily. This study not only advances our understanding about fixed points in non-standard spaces but also prepares the way for dealing with problems which cut across different fields, from pure mathematics to applied science, where set-valued mappings and generalized metrics play an important role.

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